

Mathematics: Analysis and Approaches Higher Level

Internal Assessment

**Investigation of the motion of double pendulums using  
differential equations**

Session: May 2021

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# 1 Introduction

## 1.1 Simple pendulums

One of the most basic examples of periodic motion in mechanics is the simple pendulum under a gravitational field. It is in high school physics textbooks as an everyday example of periodic motion, and in undergraduate beginner physics textbooks as a beginner's problem. It consists of a point mass on a massless string.

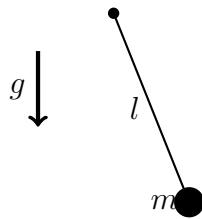


Figure 1: A simple pendulum under a gravitational field

Its equation of motion has been well-studied and is given below.

$$\ddot{\theta} = -\frac{g}{l} \sin \theta \quad (1)$$

One dot on top of the variable symbol denotes one time derivative. This notation will be used throughout this investigation.  $\theta$  is the angle between the vertical axis and the string,  $l$  is the length of the string, and  $g$  is the gravitational field strength.

At small angles, equation (1) can be changed to another form using the small angle approximation ( $\sin \theta \approx \theta$ ) (Kleppner & Kolenkow, 2014):

$$\ddot{\theta} = -\frac{g}{l} \theta \quad (2)$$

This is a case of the equation below:

$$\frac{d^2 y}{dx^2} = -\omega^2 y \quad (3)$$

This is a well-known ODE with an analytic solution. For equation (2), the solution is:

$$\theta = A \sin(\omega t + \phi), \quad \omega^2 = \frac{g}{l} \quad (4)$$

Where  $A$  and  $\phi$  depend on initial conditions.(Keisler, 2012)

If the angle of oscillation is larger, while equation (1) still has an analytic solution, it is quite complex. Most will simply obtain numerical solutions using Euler's Method.

For example, this is the numerical solution for  $\theta_0 = 0.75\pi, l = 1, g = 9.81$ , and zero initial angular velocity.

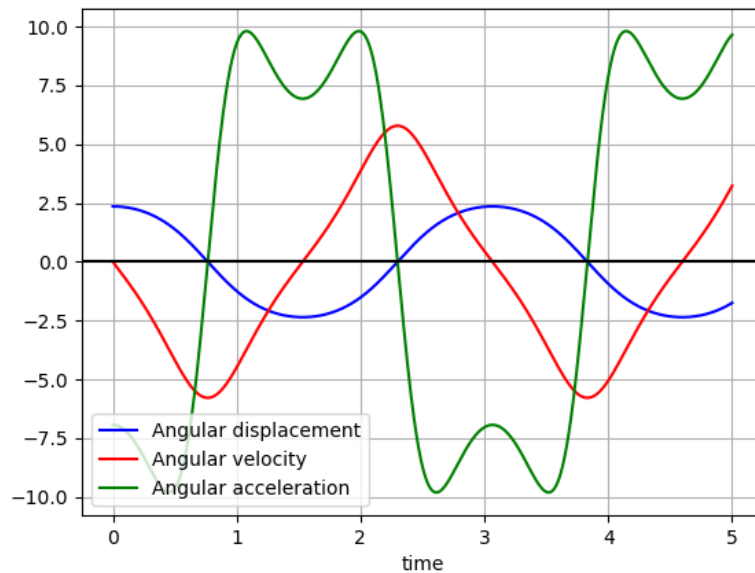


Figure 2: Numerical solution using Euler's method with a time-step of 0.0001

As a student who regularly reads materials in both mathematics and physics, I have encountered the simple pendulum, and similar periodic systems, many times. However, a much more fascinating system is the double pendulum.

## 1.2 Double pendulums

Double pendulums consist of two rigid massless rods and two point masses.

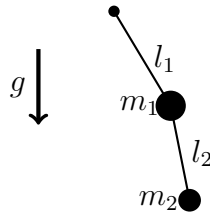


Figure 3: Double pendulum under gravitational field

The motion of double pendulum is often claimed as "unpredictable", "chaotic", and "unintuitive". It is a famous example of the "butterfly effect" in chaos theory due to the sensitivity of its motion to initial conditions (Levien & Tan, 1993). It is also well-known for being a problem that is rarely solved using Newtonian mechanics. Instead, Lagrangian mechanics (an alternative formulation of mechanics based on energy) is used.

Having an interest in mathematical modelling, I decided to attempt simulating the double pendulum using my own knowledge in physics and the mathematical skills taught in IB Mathematics HL.

## 1.3 Aim

In this investigation: I will endeavour to find out about the motion of the double pendulum by deriving the ODEs using Lagrangian mechanics, then using Euler's method to find numerical solutions of its motion.

# 2 Mathematics used: Lagrangian mechanics and partial differentiation

Lagrangian mechanics is an alternative formulation of mechanics developed by physicist and mathematician Joseph-Louis Lagrange. It is unlike Newtonian mechanics, as it is based on energy calculations, with no need of calculation using vectors.

It is based upon the Euler-Lagrange equation (seen below), which is a result from the Principle of Least Action and calculus of variations (the study of the minima and maxima of functionals) (Susskind & Hrabovsky, 2020). It serves as an alternative to  $F = ma$ .

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = \frac{\partial \mathcal{L}}{\partial q_j} \quad (5)$$

It often involves several variables, and has to use partial derivatives, denoted by  $\partial$ . This means the differentiation is focused on one variable, and all others are treated as constants.

In the equations,  $q_j$  is a generalised coordinate, and  $\mathcal{L}$  is the Lagrangian. It is defined below (Morin, 2007):

$$\mathcal{L} = T - V \quad (6)$$

$$T = \sum_{k=1}^N m_k v_k^2 \quad (7)$$

$T$  is the kinetic energy (which depends on mass  $m$  and speed  $v$ ), and  $V$  is the potential energy. The potential energy depends on the existing force. (Morin, 2007)

In the case of an object under a gravitational field:

$$V = \sum_{k=1}^N m_k g y_k \quad (8)$$

Where  $y$  is the height. Its value depends on the reference level (where potential energy is defined as zero) chosen. (Morin, 2007)

In the above equations,  $N$  is the number of objects in the system, and  $k$  is the designation given to each object.

To solve for the equations of motion, there will be multiple Euler-Lagrange equations, each based on one coordinate. For example,  $q_1$  can be  $\theta_1$ , so  $\dot{q}_1$  will have to be  $\dot{\theta}_1$ .  $\theta_2$  will be treated as a constant during differentiation and formulation of the equation.

An example of this partial differentiation can be demonstrated by a simple multivariable function:

$$f(\theta_1, \theta_2) = \theta_1^2 + \theta_1\theta_2 + \theta_2^2$$

$$\frac{\partial f}{\partial \theta_1} = 2\theta_1 + \theta_2 \quad , \quad \frac{\partial f}{\partial \theta_2} = \theta_1 + 2\theta_2$$

$$\frac{d}{dt} \frac{\partial f}{\partial \theta_1} = 2\dot{\theta}_1 + \dot{\theta}_2 \quad , \quad \frac{d}{dt} \frac{\partial f}{\partial \theta_2} = \dot{\theta}_1 + 2\dot{\theta}_2$$

### 3 The equations of motion

#### 3.1 Description of the system

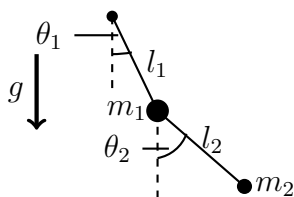


Figure 4: Diagram of double pendulum with all parameters used

Refer to figure 4 for definitions of parameters.  $l_1$ ,  $l_2$ ,  $m_1$ , and  $m_2$  are constant.

For the frame of reference, the top fixed pivot will be used as the origin.

For the equations of motion,  $\theta_1$  and  $\theta_2$  will be solved using the Euler-Lagrange equation.

$x_1$ ,  $x_2$ ,  $y_1$ , and  $y_2$  can be found using the below equations:

$$x_1 = l_1 \sin \theta_1 \tag{9}$$

$$y_1 = -l_1 \cos \theta_1 \tag{10}$$

$$x_2 = x_1 + l_2 \sin \theta_2 \tag{11}$$

$$y_2 = y_1 - l_2 \cos \theta_2 \tag{12}$$

To list the required Euler-Lagrange equations, the kinetic and potential energies will need to be computed first to assemble the Lagrangian. They are computed below:

**Kinetic energy** For computation of kinetic energy,  $v^2$  must first be expressed in terms of given variables

Using Pythagoras' Theorem ( $v^2 = \dot{x}^2 + \dot{y}^2$ ) and equation (7):

$$T_1 = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) \quad (13)$$

$$T_2 = \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \quad (14)$$

Expanding using equations (9) to (12):

$$T_1 = \frac{m_1}{2}l_1^2\dot{\theta}_1^2 \quad (15)$$

$$T_2 = m_2\left[\frac{1}{2}l_1^2\dot{\theta}_1^2 + \frac{1}{2}l_2^2\dot{\theta}_2^2 + l_1l_2 \cos(\theta_2 - \theta_1)\dot{\theta}_1\dot{\theta}_2\right] \quad (16)$$

**Gravitational potential energies** Using equation (8):

$$V_1 = m_1gy_1 \quad (17)$$

$$V_2 = m_2gy_2 \quad (18)$$

Expanding using equation (10) and (12):

$$V_1 = -m_1gl_1 \cos \theta_1 \quad (19)$$

$$V_2 = -m_2g(l_1 \cos \theta_1 + l_2 \cos \theta_2) \quad (20)$$

**The Lagrangian** Using the definition of the Lagrangian (equation 6) and the energies derived above:

$$\begin{aligned} \mathcal{L} &= T_1 + T_2 - V_1 - V_2 \\ &= m_1\left(\frac{1}{2}l_1^2\dot{\theta}_1^2 + gl_1 \cos \theta_1\right) \\ &\quad + m_2\left[\frac{1}{2}l_1^2\dot{\theta}_1^2 + \frac{1}{2}l_2^2\dot{\theta}_2^2 + l_1l_2 \cos(\theta_2 - \theta_1)\dot{\theta}_1\dot{\theta}_2 + gl_1 \cos \theta_1 + gl_2 \cos \theta_2\right] \end{aligned} \quad (21)$$



### 3.2 The Euler-Lagrange equations of the system

To get the equation of motion, one can simply use the Euler-Lagrange equation (eq. 5) For this system, we can use two Euler-Lagrange equations, where  $q_1$  is  $\theta_1$  and  $q_2$  is  $\theta_2$ :

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = \frac{\partial \mathcal{L}}{\partial \theta_1} \quad (22)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = \frac{\partial \mathcal{L}}{\partial \theta_2} \quad (23)$$

Then, finally, we can derive the equations of motion. Differentiating, we get:

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} &= (m_1 + m_2)l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 [\ddot{\theta}_2 \cos(\theta_2 - \theta_1) - \dot{\theta}_2 (\dot{\theta}_2 - \dot{\theta}_1) \sin(\theta_2 - \theta_1)] \\ &= \frac{\partial \mathcal{L}}{\partial \theta_1} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) - (m_1 + m_2) g l_1 \sin \theta_1 \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} &= m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 [\ddot{\theta}_1 \cos(\theta_2 - \theta_1) - \dot{\theta}_1 (\dot{\theta}_2 - \dot{\theta}_1) \sin(\theta_2 - \theta_1)] \\ &= \frac{\partial \mathcal{L}}{\partial \theta_2} = -m_2 l_2 [l_1 \sin(\theta_2 - \theta_1) \dot{\theta}_1 \dot{\theta}_2 + g \sin \theta_2] \end{aligned} \quad (25)$$

These equations can then be transformed into a form suitable for numerical solving:

$$\begin{aligned} [1 - \frac{m_2}{m_1 + m_2} \cos^2(\theta_2 - \theta_1)] l_1 \ddot{\theta}_1 &= -g \sin \theta_1 + \frac{m_2}{m_1 + m_2} \sin(\theta_2 - \theta_1) (l_2 \dot{\theta}_2^2 + l_1 \cos(\theta_2 - \theta_1) \dot{\theta}_1^2) \\ &\quad + \frac{m_2}{m_1 + m_2} g \cos(\theta_2 - \theta_1) \sin \theta_2 \end{aligned} \quad (26)$$

$$\begin{aligned} [1 - \frac{m_2}{m_1 + m_2} \cos^2(\theta_2 - \theta_1)] l_2 \ddot{\theta}_2 &= -g \sin \theta_2 - \sin(\theta_2 - \theta_1) \dot{\theta}_1^2 \\ &\quad - \cos(\theta_2 - \theta_1) [\frac{m_2 l_2}{m_1 + m_2} \sin(\theta_2 - \theta_1) \dot{\theta}_2^2 - g \sin \theta_1] \end{aligned} \quad (27)$$

One method for verifying these equations is to take the limiting case of the single pendulum, which means letting  $m_2$  and  $l_2$  approach 0. In this case, equation (26) becomes:

$$l_1 \ddot{\theta}_1 = -g \sin \theta_1$$

which is identical to equation (1).

## 4 Solving the differential equation

### 4.1 Use of Euler's Method

To numerically solve for  $\theta_1(t)$  and  $\theta_2(t)$ , Euler's method must be used. This is achieved by the equations below:

$$\theta_i(0) = \theta_{i0} \text{ (initial conditions)} \quad (28)$$

$$\omega_i = \dot{\theta}_i = \frac{d\theta_i}{dt} = f(\theta_1, \theta_2) \quad (29)$$

$$\theta_i(t + dt) = \theta_i(t) + \dot{\theta}_i(t) \times dt, \quad i = 1, 2 \quad (30)$$

However,  $\dot{\theta}_i(t)$  must also be solved numerically using Euler's method:

$$\dot{\theta}_i(0) = \omega_{i0} \text{ (initial conditions)} \quad (31)$$

$$\ddot{\theta}_i = \frac{d^2\theta_i}{dt^2} = f(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \quad (32)$$

$$\dot{\theta}_i(t + dt) = \dot{\theta}_i(t) + \ddot{\theta}_i(t) \times dt, \quad i = 1, 2 \quad (33)$$

$\ddot{\theta}_1$  and  $\ddot{\theta}_2$  can be found by using equations (26) and (27). They are used in equation (33) to solve for  $\dot{\theta}_1$  and  $\dot{\theta}_2$ , which are then used in equation (30) to find angular displacements.

### 4.2 Example of solving the DEs using Euler's Method

**Conditions:**

$$m_1 = m_2 = l_1 = l_2 = 1$$

$$\theta_1(0) = 0.25\pi, \quad \theta_2(0) = 0.5\pi$$

$$\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$$

$$dt = 0.001$$

$$\left[1 - \frac{1}{2} \cos^2(0.25\pi)\right] \ddot{\theta}_1(0) = -g \sin(0.25\pi) + \frac{1}{2} g \cos(0.25\pi) \sin(0.5\pi)$$

$$\ddot{\theta}_1(0) = -\frac{\sqrt{2}}{3} g$$

$$\left[1 - \frac{1}{2} \cos^2(0.25\pi)\right] \ddot{\theta}_2(0) = -g \sin(0.5\pi) - \cos(0.25\pi)[-g \sin(0.25\pi)]$$

$$\ddot{\theta}_2(0) = -\frac{2}{3} g$$

$$\dot{\theta}_1(0.001) = \dot{\theta}_1(0) + 0.001 \times \ddot{\theta}_1(0)$$

$$= -0.001 \times \frac{\sqrt{2}}{3} \times 9.81 = -4.62 \times 10^{-3} s^{-1}$$

$$\theta_1(0.001) = \theta_1(0) + 0.001 \times \dot{\theta}_1(0)$$

$$= 0.25\pi$$

$$\theta_1(0.002) = \theta_1(0.001) + 0.001 \times \dot{\theta}_1(0.001)$$

$$= 0.25\pi - 4.62 \times 10^{-6}$$


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$$\dot{\theta}_2(0.001) = \dot{\theta}_2(0) + 0.001 \times \ddot{\theta}_2(0)$$

$$= -0.001 \times \frac{2}{3} \times 9.81 = -6.54 \times 10^{-3} s^{-1}$$

$$\theta_2(0.001) = \theta_2(0) + 0.001 \times \dot{\theta}_2(0)$$

$$= 0.5\pi$$

$$\theta_2(0.002) = \theta_2(0.001) + 0.001 \times \dot{\theta}_2(0.001)$$

$$= 0.5\pi - 6.54 \times 10^{-6}$$

As these equations are complicated, and there are many iterations, it is difficult to do by hand. It is better to use computer software. I chose to use the programming language Python, with the libraries math, numpy, and matplotlib.

### 4.3 Trials

Common results for analysis of double pendulum include trajectories of the masses, and graphs of the angular displacements as functions of time.

A chaotic nature means small changes in initial conditions bring about a significant difference in final results, which is the "butterfly effect". This can be verified by doing trials with slightly different initial conditions.

All trials are done with  $dt = 0.0001$  and  $g = 9.81$ .  $\theta$  is in radians. T is the duration of the trial.

Description	$l_1$	$l_2$	$m_1$	$m_2$	$\theta_{10}$	$\omega_{10}$	$\theta_{20}$	$\omega_{20}$	T
Standard	1	1	1	1	$0.25\pi$	0	$0.75\pi$	0	20
5% increase in $\theta_{10}$	1	1	1	1	$0.2625\pi$	0	$0.75\pi$	0	20
5% increase in $\theta_{20}$	1	1	1	1	$0.25\pi$	0	$0.7875\pi$	0	20
Small initial $\omega_{10}$	1	1	1	1	$0.25\pi$	$-0.01\pi$	$0.75\pi$	0	20
Small initial $\omega_{20}$	1	1	1	1	$0.25\pi$	0	$0.75\pi$	$-0.01\pi$	20
5% increase in $l_1$	1.05	1	1	1	$0.25\pi$	0	$0.75\pi$	0	20
5% increase in $l_2$	1	1.05	1	1	$0.25\pi$	0	$0.75\pi$	0	20
5% increase in $m_1$	1	1	1.05	1	$0.25\pi$	0	$0.75\pi$	0	20
5% increase in $m_2$	1	1	1	1.05	$0.25\pi$	0	$0.75\pi$	0	20
Approximation to simple pendulum: Small angle	1	0.0001	1	0.0001	$0.05\pi$	0	0	0	5
Approximation to simple pendulum: Large angle	1	0.0001	1	0.0001	$0.75\pi$	0	0	0	5

The standard trial serves as a control to compare against all other trials. Its initial condition is represented in the diagram below.

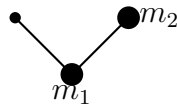
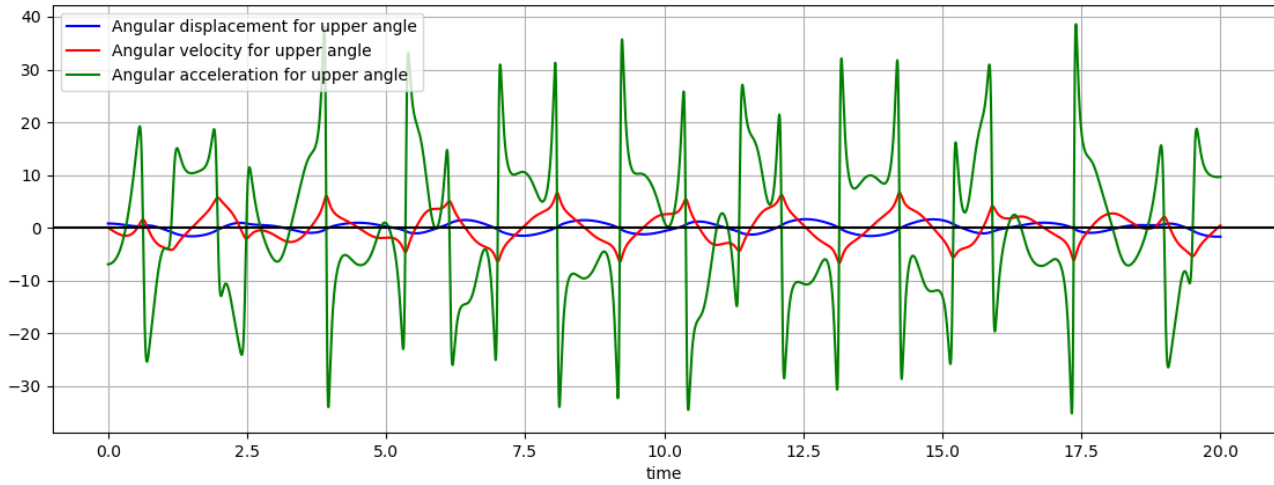


Figure 5: Initial conditions of standard trial

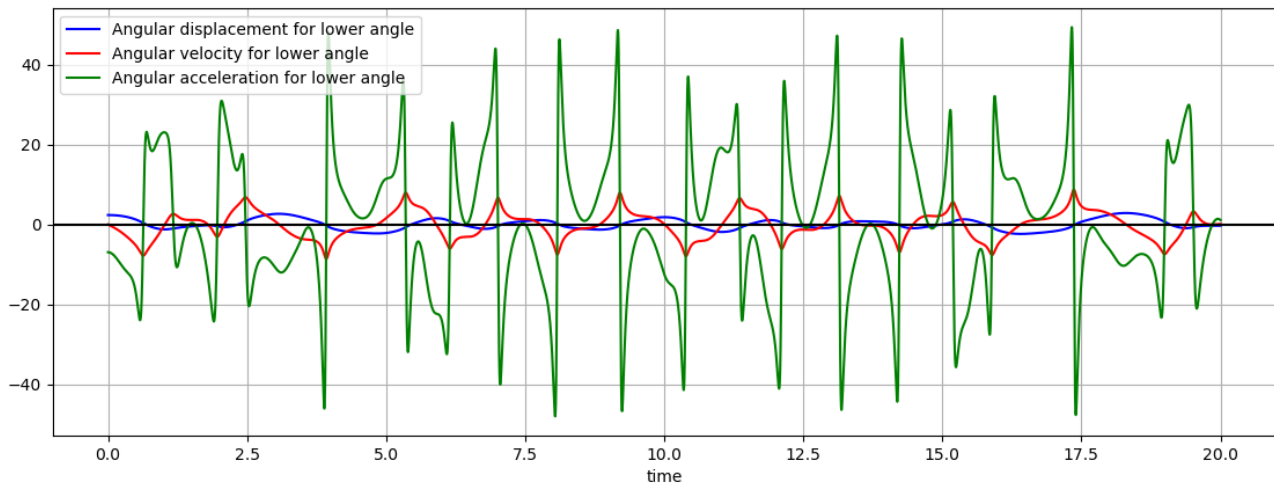
# 5 Results

## 5.1 Behaviour of standard double pendulum

The following are all results of the standard trial.



(a) Behaviour of  $\theta_1$



(b) Behaviour of  $\theta_2$

Figure 6: Behaviour of angles in standard double pendulum

There are several evident properties for all parameters seen in the graphs:

- There is no clear period, amplitude, or wavelength
- There are sharp maxima and minima
- There is no clear relationship between behaviours of the two masses

These properties can also be seen in online demonstrations of double pendulums. (Neumann, 2002)

The trajectory was calculated by finding  $x_1, x_2, y_1, y_2$  at each moment using equations (9) to (12).

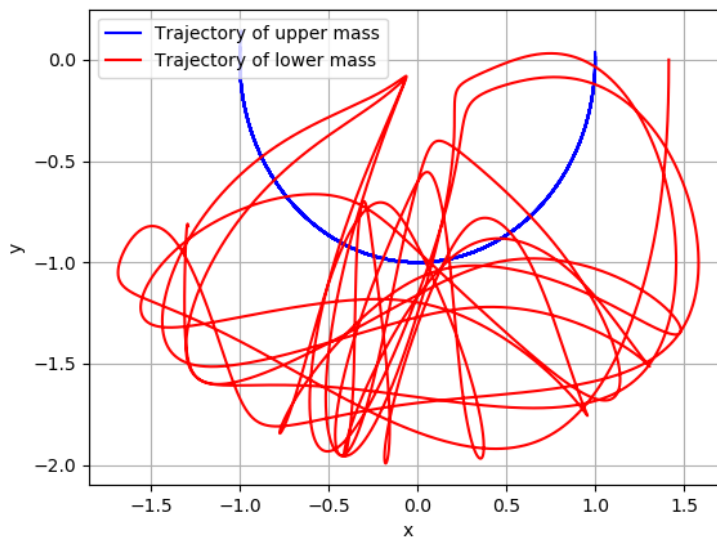


Figure 7: Trajectory of standard pendulum

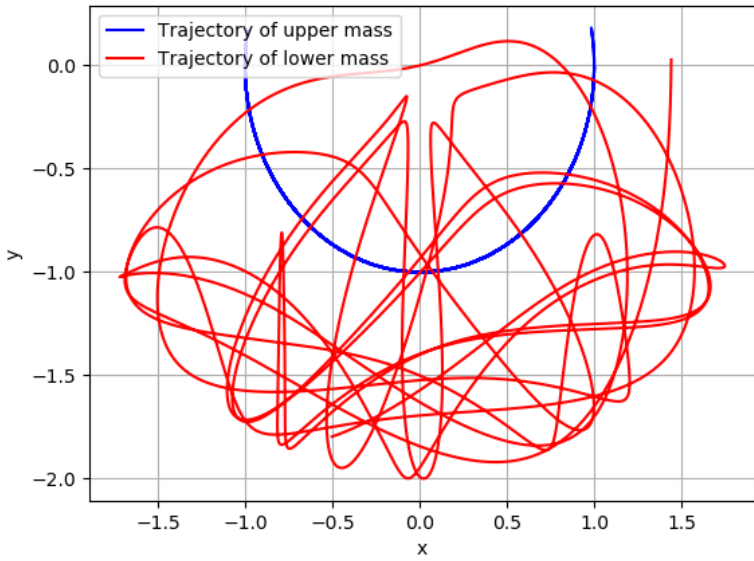
Again, multiple conclusions can be drawn:

- The upper mass sweeps out an asymmetrical, continuous circular arc (the speed is inconsistent, which can't be seen in the trajectory)
- The motion of the lower mass has no clear pattern. It can form large loops or sharp corners.
- Both masses can reach a height above the original

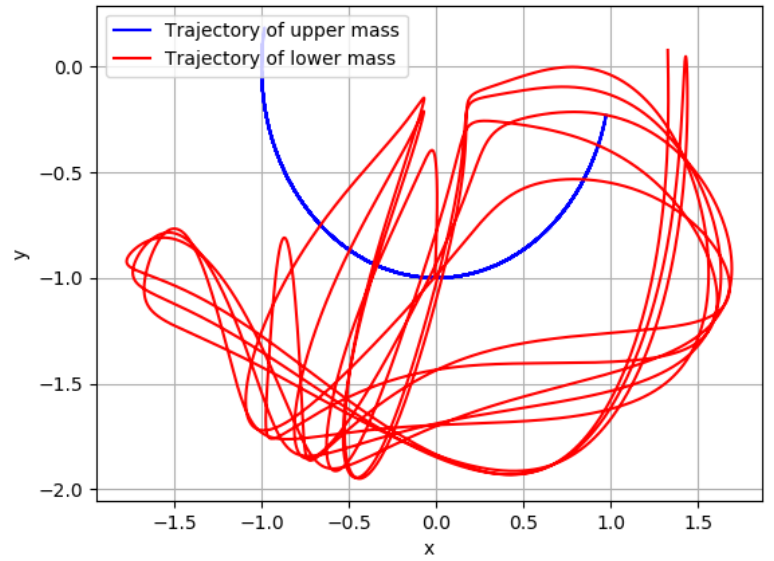
The last property can be explained via energy conservation. As the one falls and loses energy, the other gains energy and can get back up to or over its initial height. This allows the lower mass to form large loops. This phenomenon continues throughout all other trials.

## 5.2 Behaviour of the double pendulum after tweaks in initial conditions

It is easiest to compare between trials by looking at trajectories, as they give a better understanding of the movement of the pendulum. Graphs of the angular parameters will be given in the appendix.

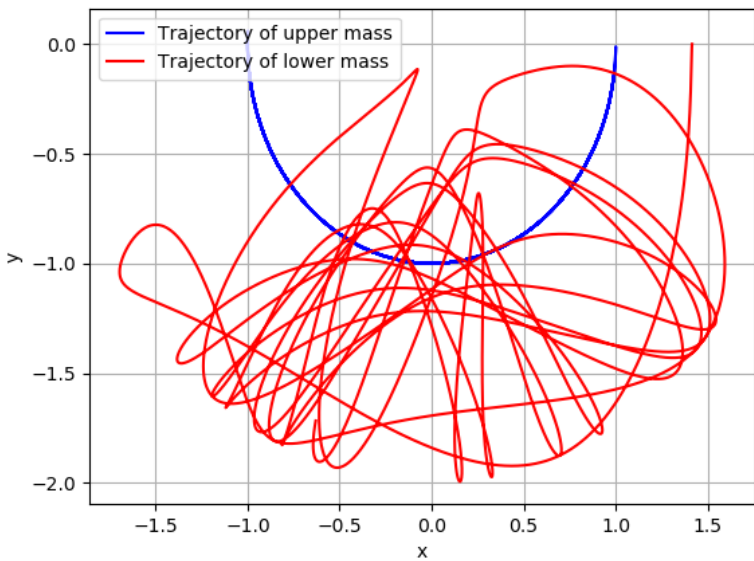


(a) Trajectory after 5% increase in  $\theta_{10}$

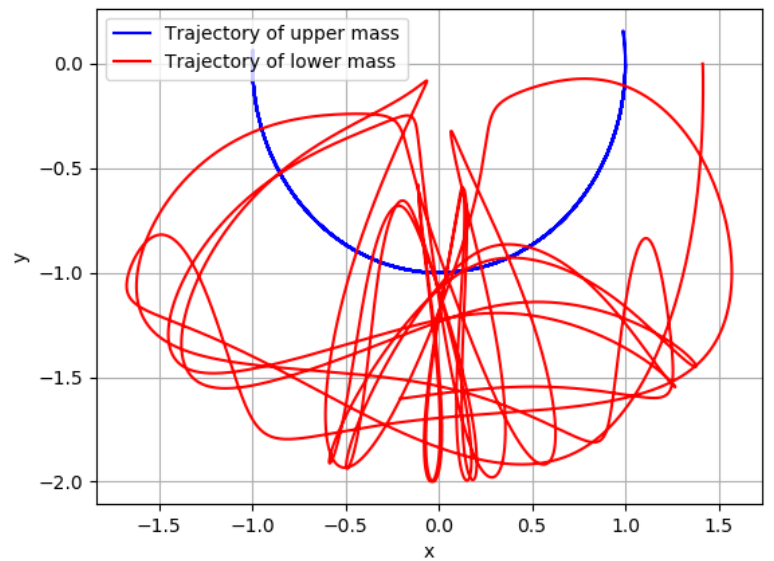


(b) Trajectory after 5% increase in  $\theta_{20}$

Figure 8: Trajectories after changes in initial angle

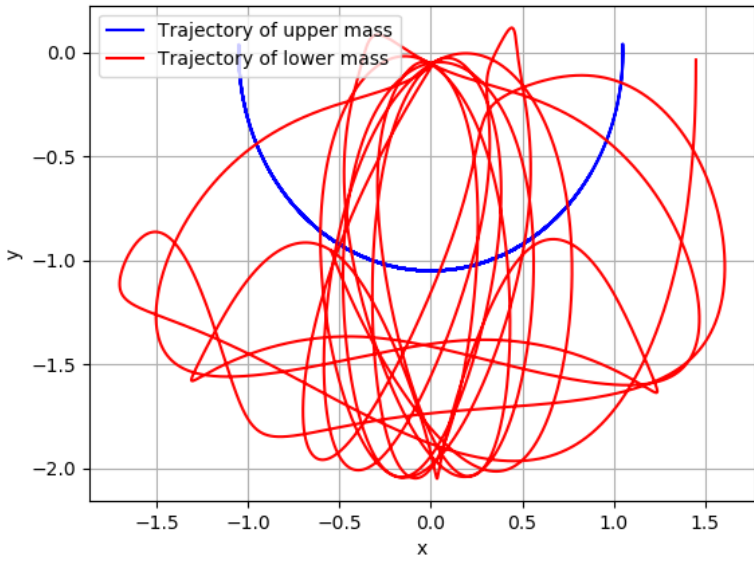


(a) Trajectory after a small initial  $\omega_{10}$

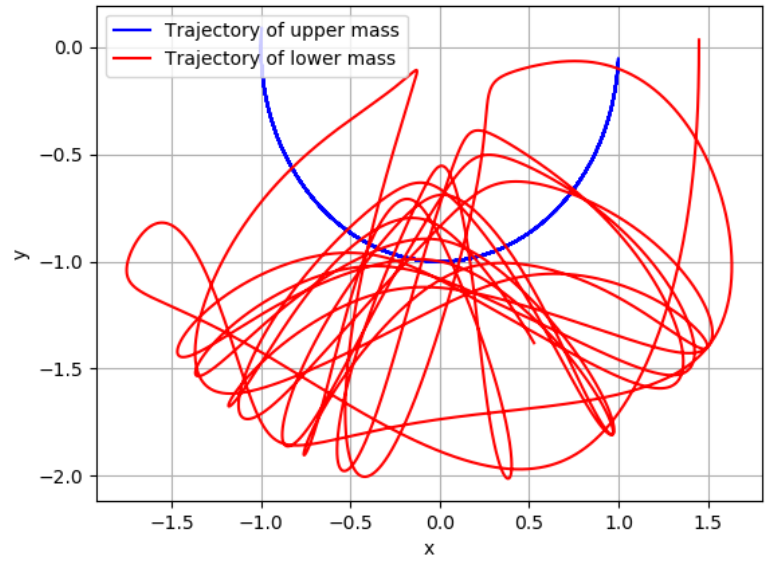


(b) Trajectory after a small initial  $\omega_{20}$

Figure 9: Trajectories after pendulum is given small initial angular velocity

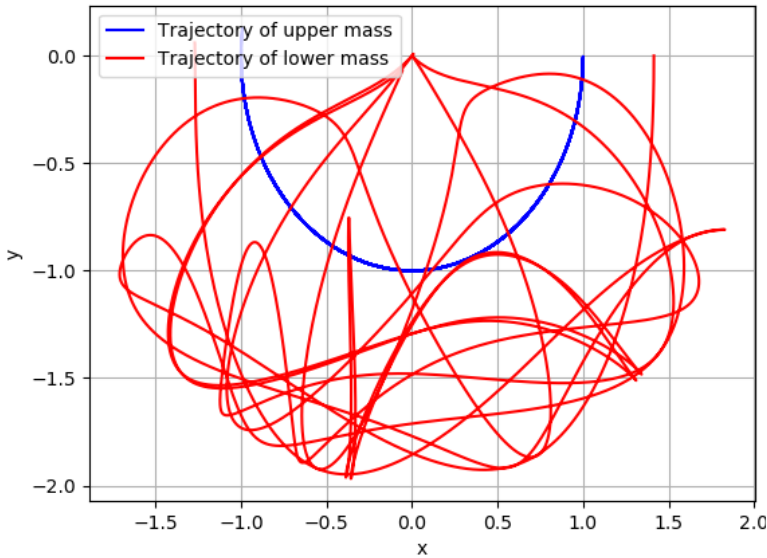


(a) Trajectory after 5% increase in  $l_1$

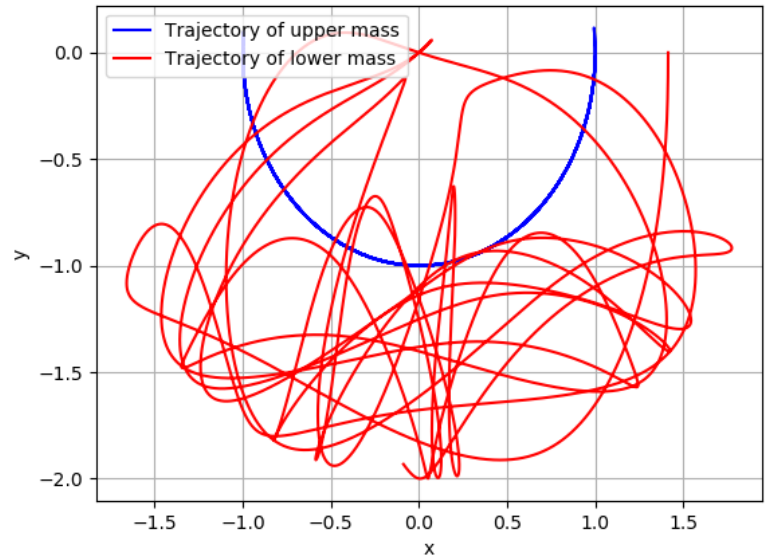


(b) Trajectory after 5% increase in  $l_2$

Figure 10: Trajectories after changes in length



(a) Trajectory after 5% increase in  $m_1$



(b) Trajectory after 5% increase in  $m_2$

Figure 11: Trajectories after changes in masses

Any change in initial conditions have resulted in wild and unpredictable changes in the movement of the masses. These include changes in positions, shapes, and sizes of the loops formed, and also the maximum heights of both masses.

The path of the lower mass is often spread out over a large area, but can also be more localised (as seen in figure 8b). The amplitude of  $\theta_1$  can also vary wildly.



### 5.3 Divergence of trajectories

It seems that variations in the trajectory manifest after some time. Before a certain time, the paths from different initial conditions are visibly similar.

This can be seen in the graphs of the angular parameters in the appendix. It can also be shown after running a trial with a shorter duration.

I ran a trial with the same initial conditions as the standard trial, except that  $T = 7.5s$

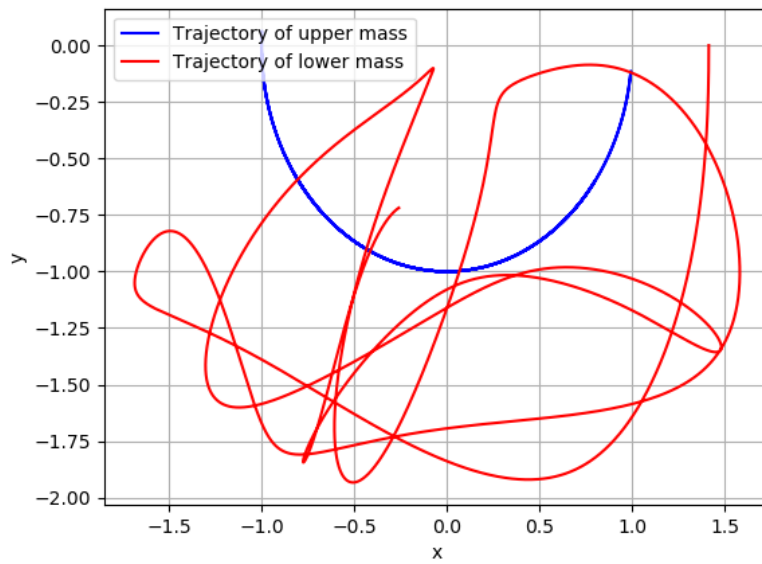


Figure 12: Trajectory in the initial 7.5 seconds

This initial pattern can be seen in the results of all trials. Although the precise path is different, the overall shape is similar. It is not until later that the trajectories begin to diverge significantly.

## 5.4 Trajectory of approximations to simple pendulum

Two trials were done with parameters chosen such that the upper mass will behave as if it is a simple pendulum, meaning that  $m_2, l_2 \ll 1$ . If the initial angle is small, it should behave according to equation (4). If the initial angle is large, it should behave similarly to figure 2.

The behaviour of the lower mass is not shown as it gives little physical insight, apart from the fact that it circulates around the upper mass at a very high angular velocity.

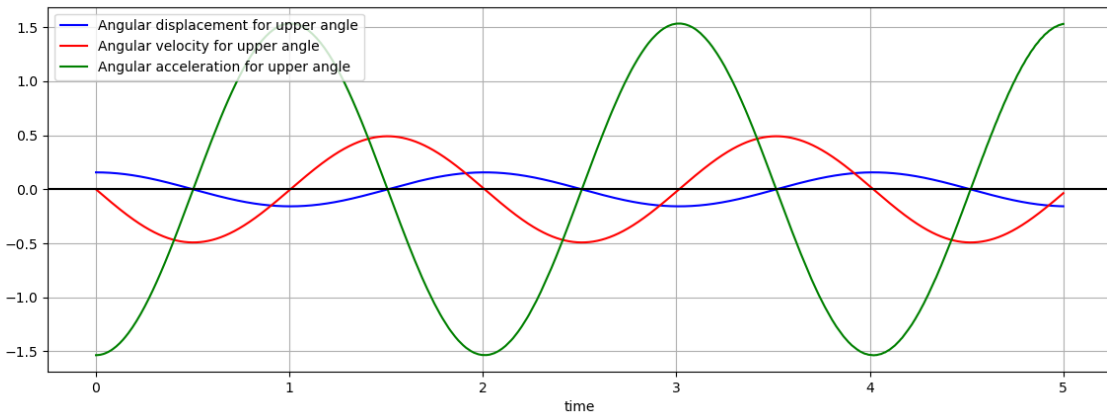


Figure 13: Behaviour of approximated single pendulum at low amplitude

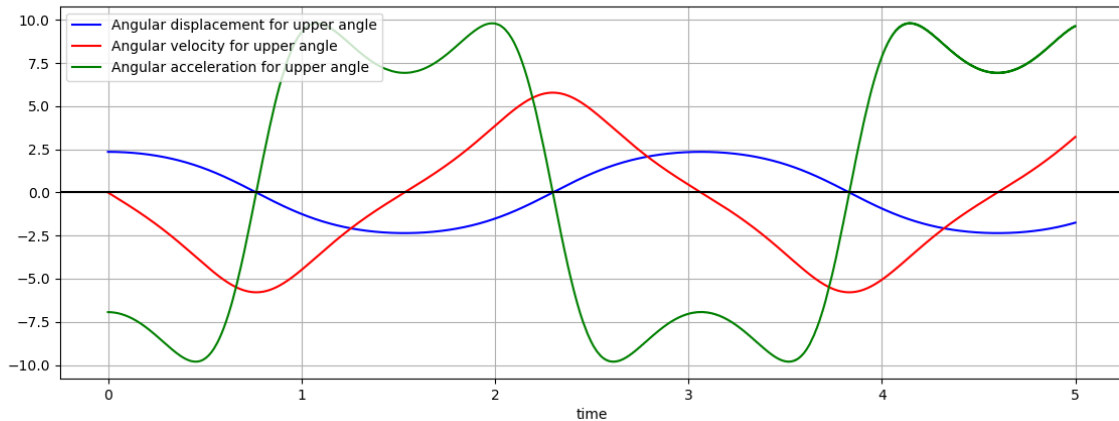


Figure 14: Behaviour of approximated single pendulum at high amplitude

As expected, at low amplitude,  $\theta_1(t)$  can be approximated by a sinusoidal function, and at high amplitude, it behaves like Figure 2. This further verifies the validity of this mathematical model.

## 6 Potential further study

**Physical pendulums** As with many other examples in physics, the simple pendulum is only an approximation. In real life, the rods would have mass, and the masses have volume. Models taking this into account are called physical pendulums. These situations are more complicated, but they can still be modelled using other equations.

**Improvement of numerical solving** For more precision, and to model even more complicated differential equations, Euler's Method may not be sufficient.

There are other, more precise methods, such as the Runge-Kutta methods, which uses more steps to make a better approximation.

## 7 Conclusion

Using Lagrangian mechanics and Euler's Method, various properties of the double pendulum have been found:

- Its behaviour is unpredictable and non-periodic
- The motion of the masses is often spread out over a large area
- Any small change in initial conditions will result in a vastly different trajectory, but only after some time
- When  $m_2, l_2 \ll 1$ , the upper mass behaves similarly to a simple pendulum

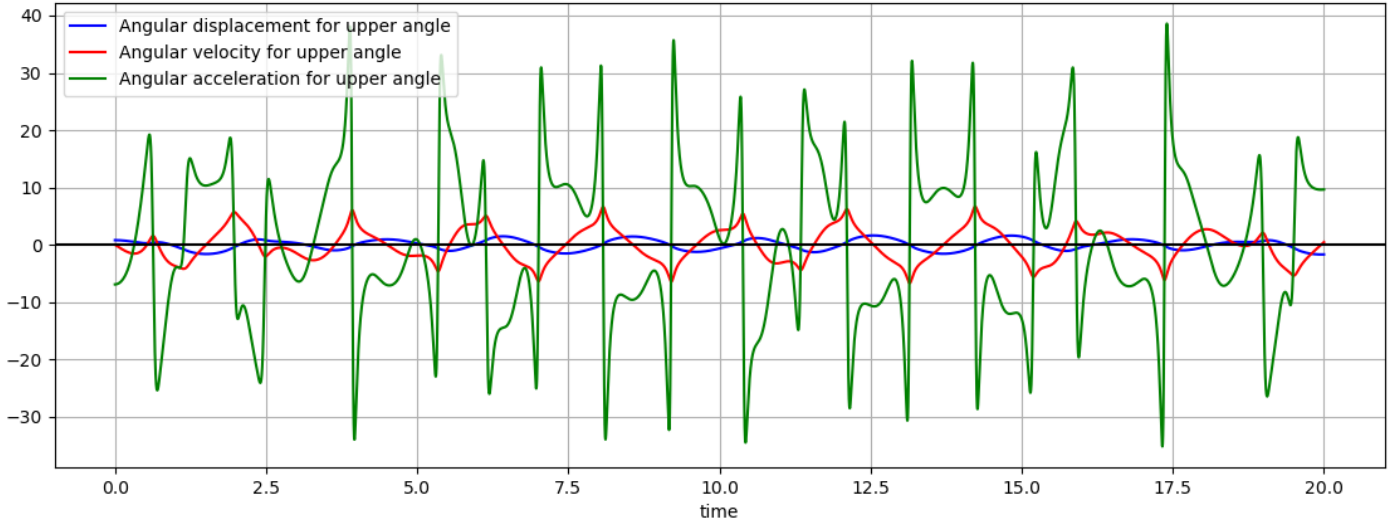
For further study, physical pendulums can be studied to match a real-life situation. Another method for solving DEs may need to be used.

## References

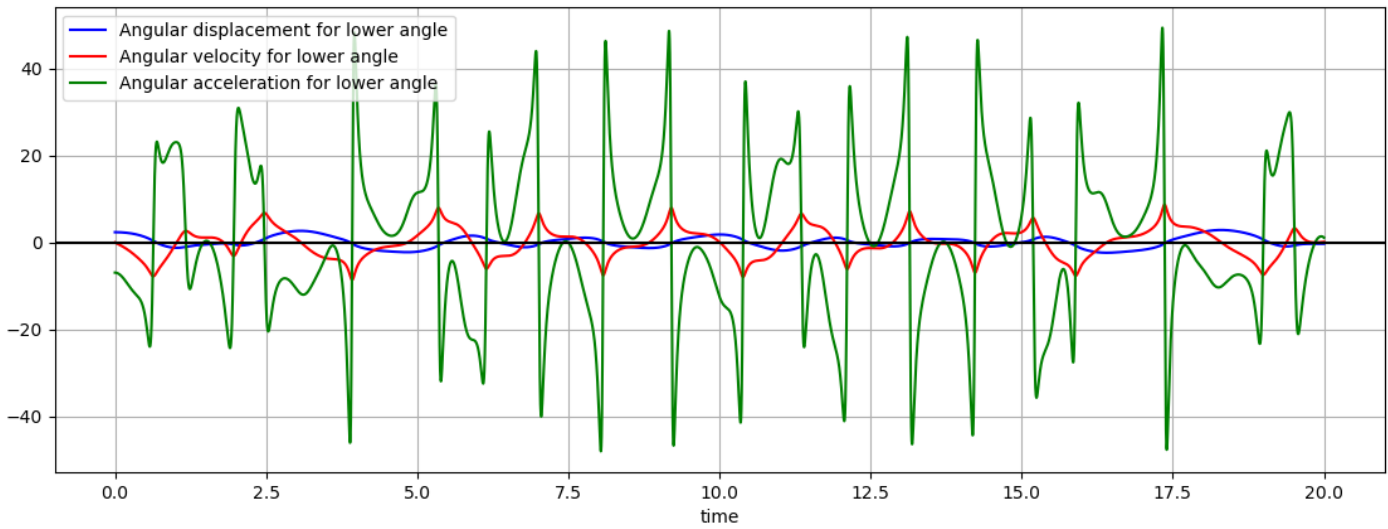
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# A Behaviour of angles in double pendulums

## A.1 Behaviour of standard double pendulum

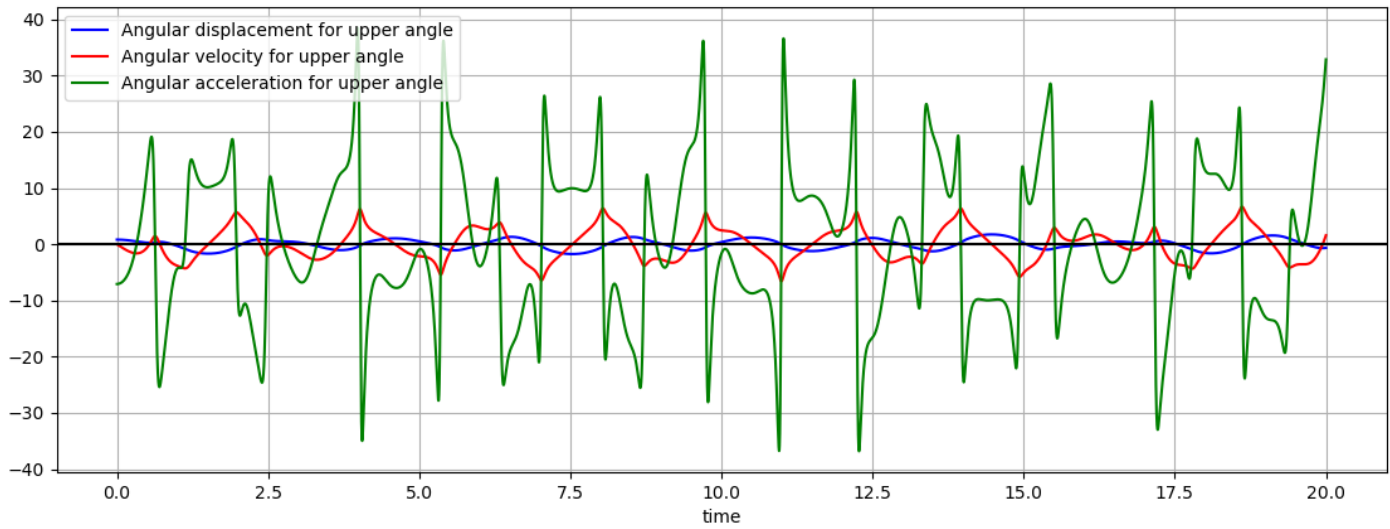


(a) Behaviour of  $\theta_1$

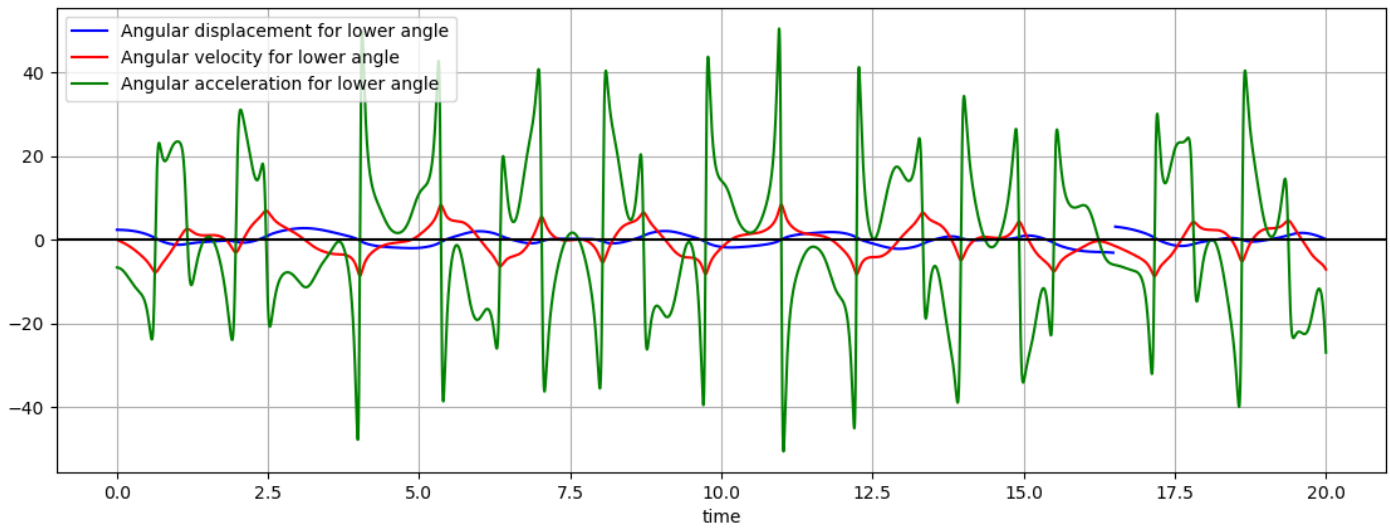


(b) Behaviour of  $\theta_2$

## A.2 Behaviour after 5% increase in $\theta_{10}$

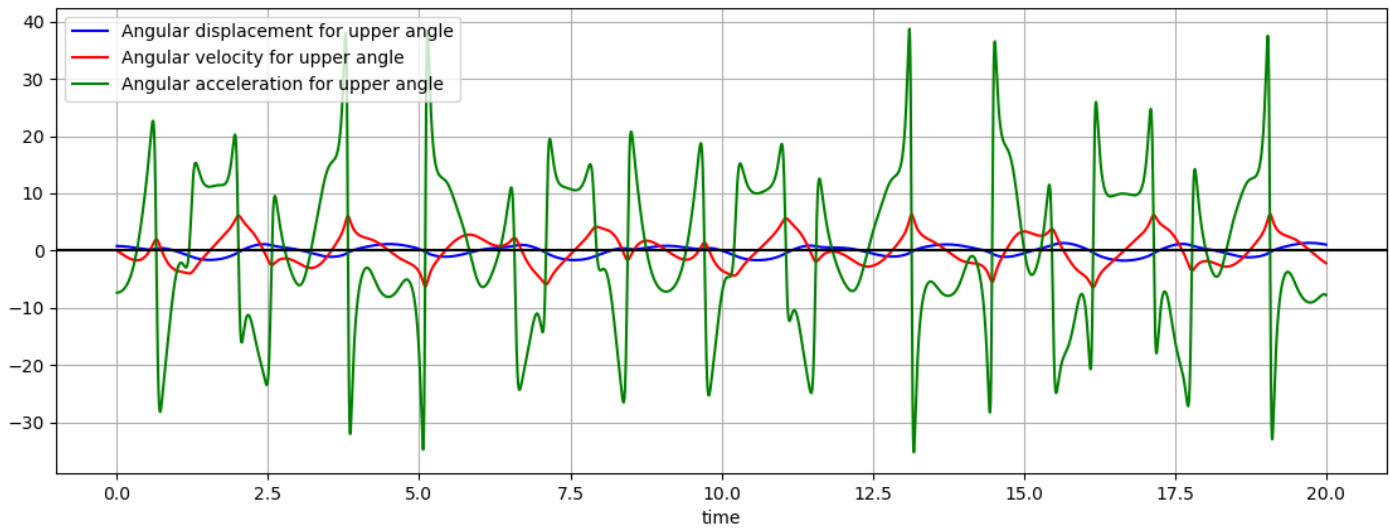


(a) Behaviour of  $\theta_1$

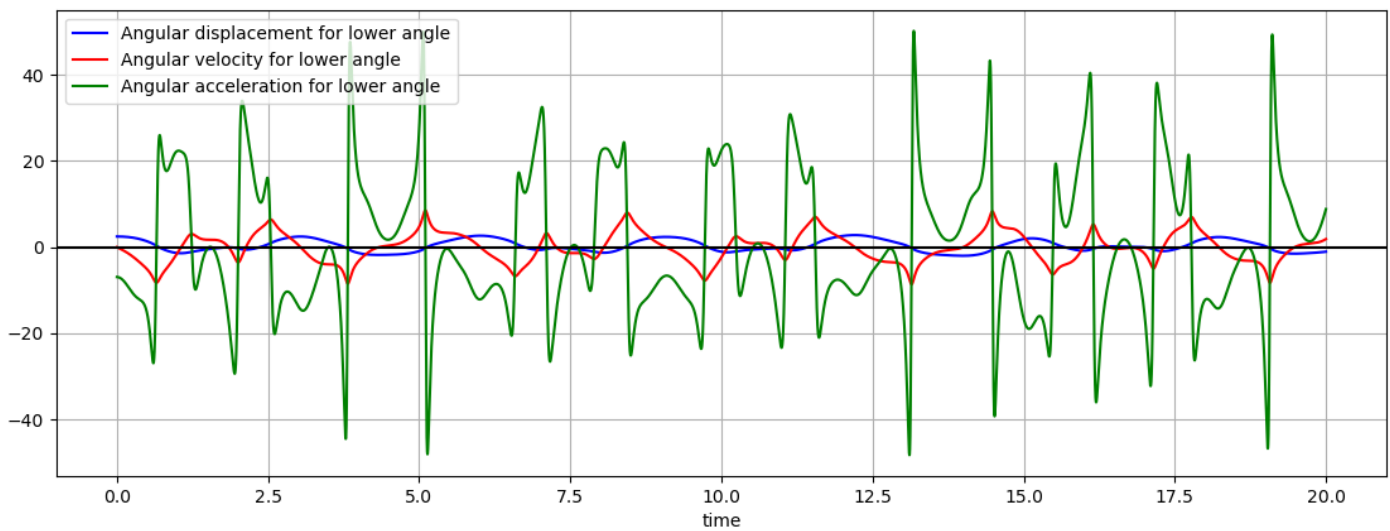


(b) Behaviour of  $\theta_2$

### A.3 Behaviour after 5% increase in $\theta_{20}$

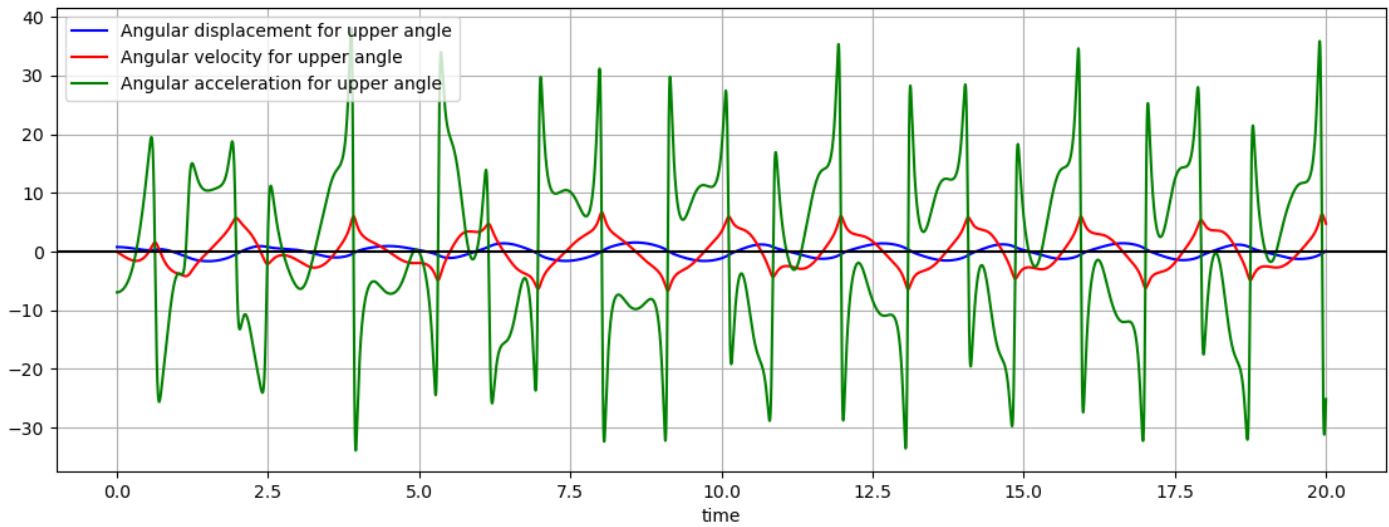


(a) Behaviour of  $\theta_1$

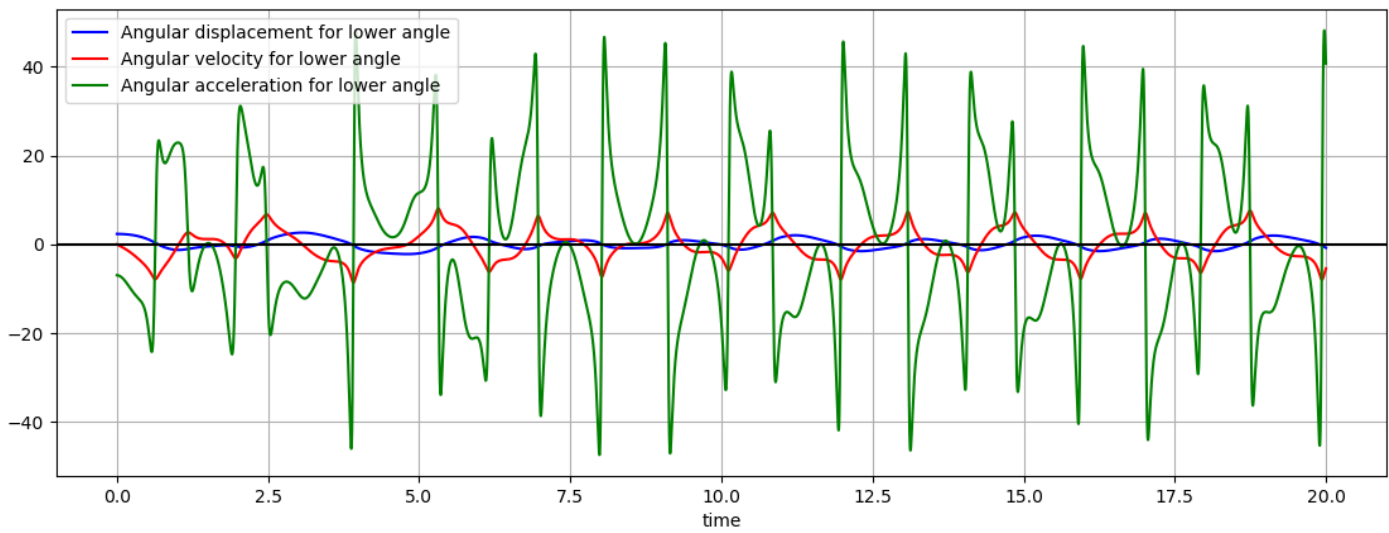


(b) Behaviour of  $\theta_2$

## A.4 Behaviour after small initial $\omega_{10}$



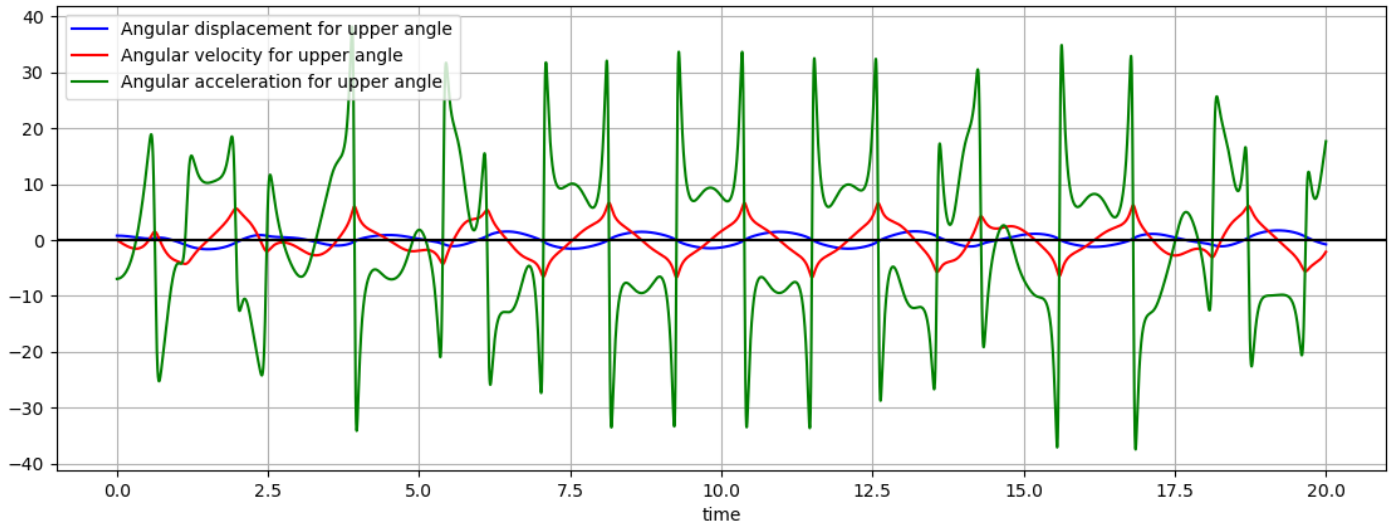
(a) Behaviour of  $\theta_1$



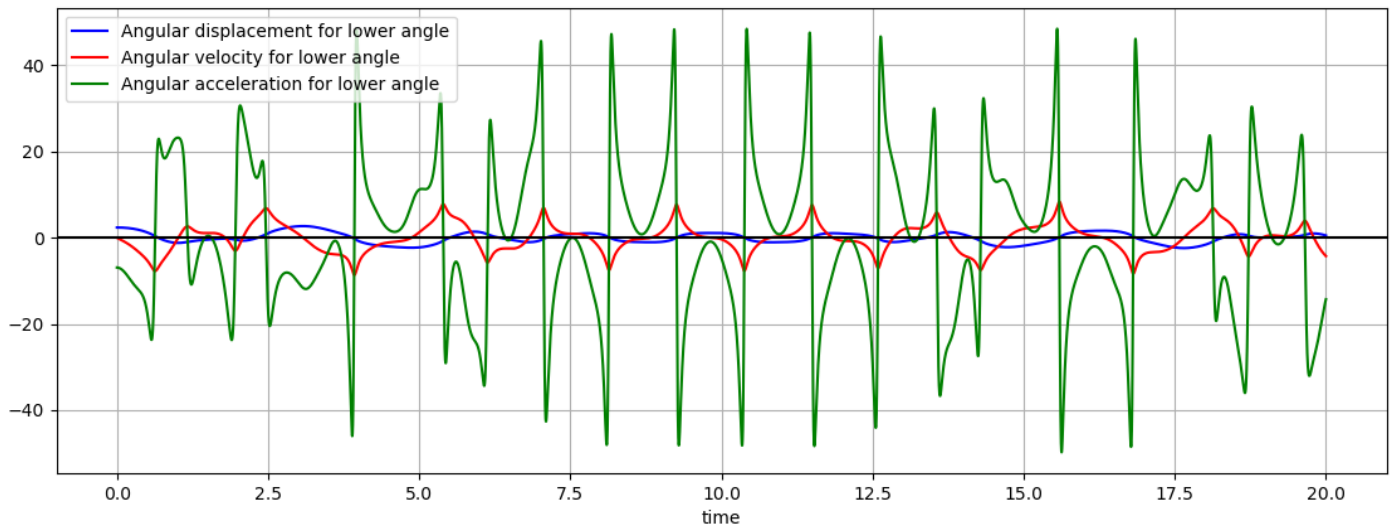
(b) Behaviour of  $\theta_2$



## A.5 Behaviour after small initial $\omega_{20}$

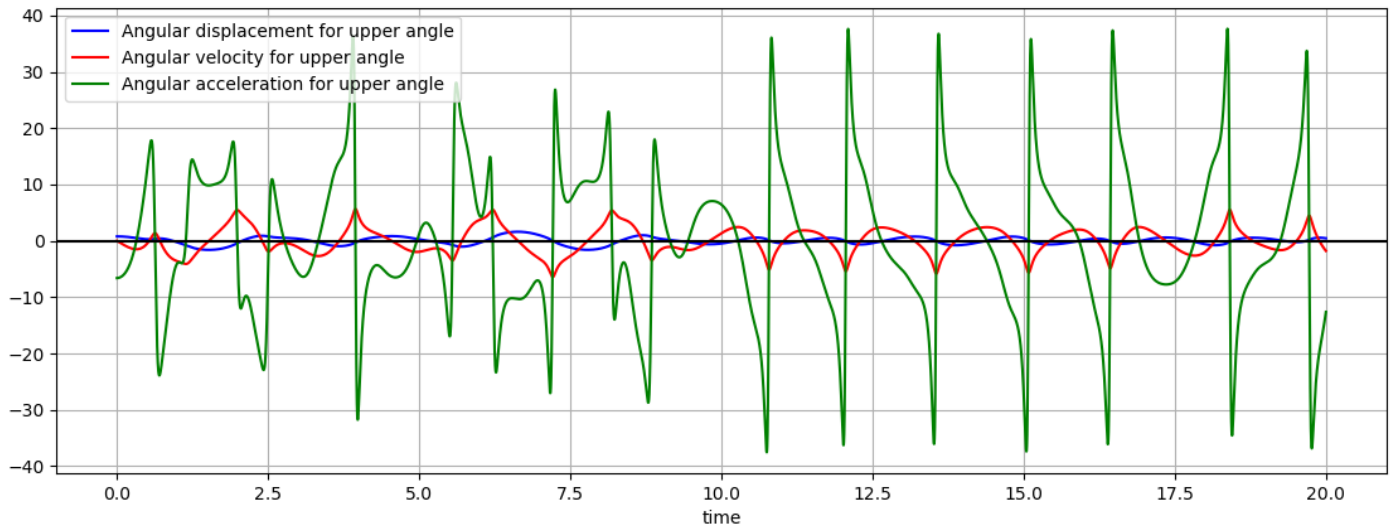


(a) Behaviour of  $\theta_1$

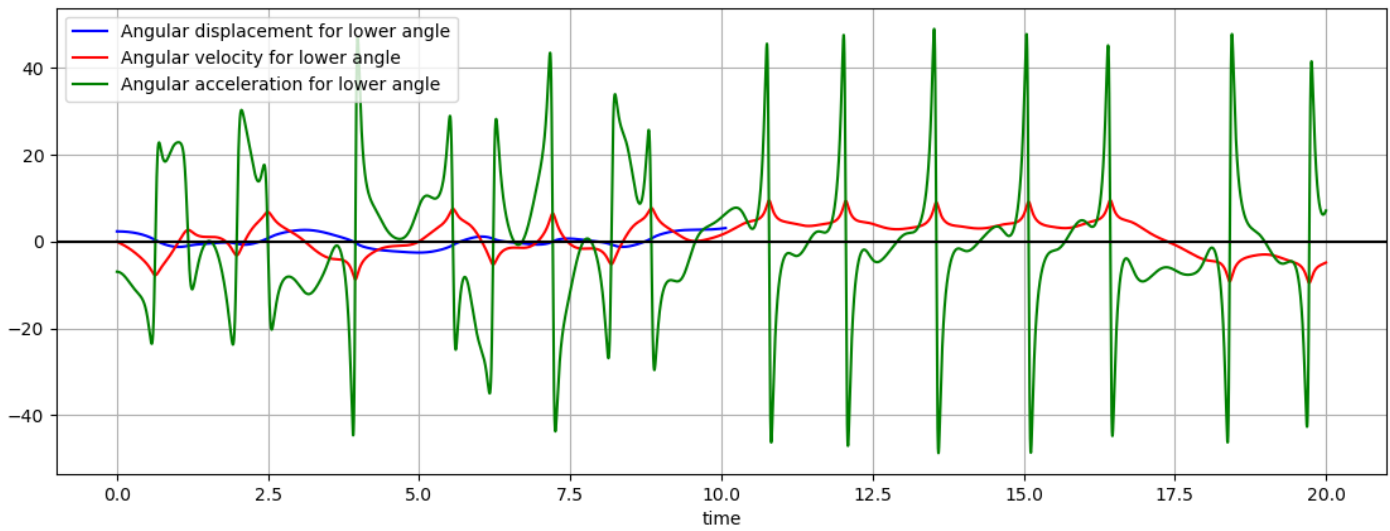


(b) Behaviour of  $\theta_2$

## A.6 Behaviour after 5% increase in $m_1$

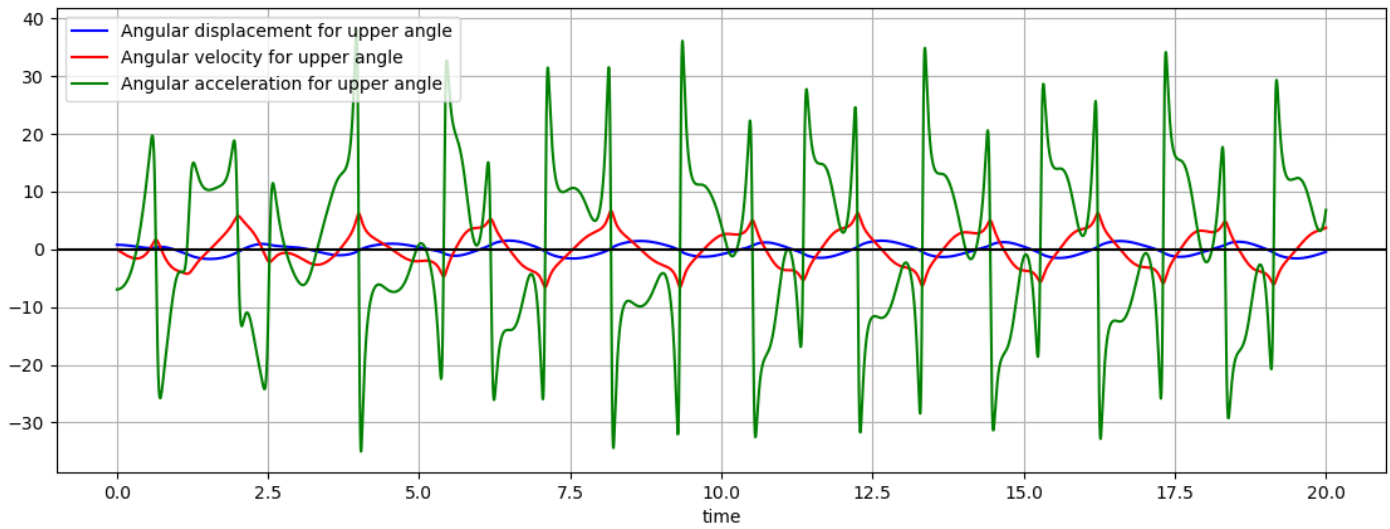


(a) Behaviour of  $\theta_1$

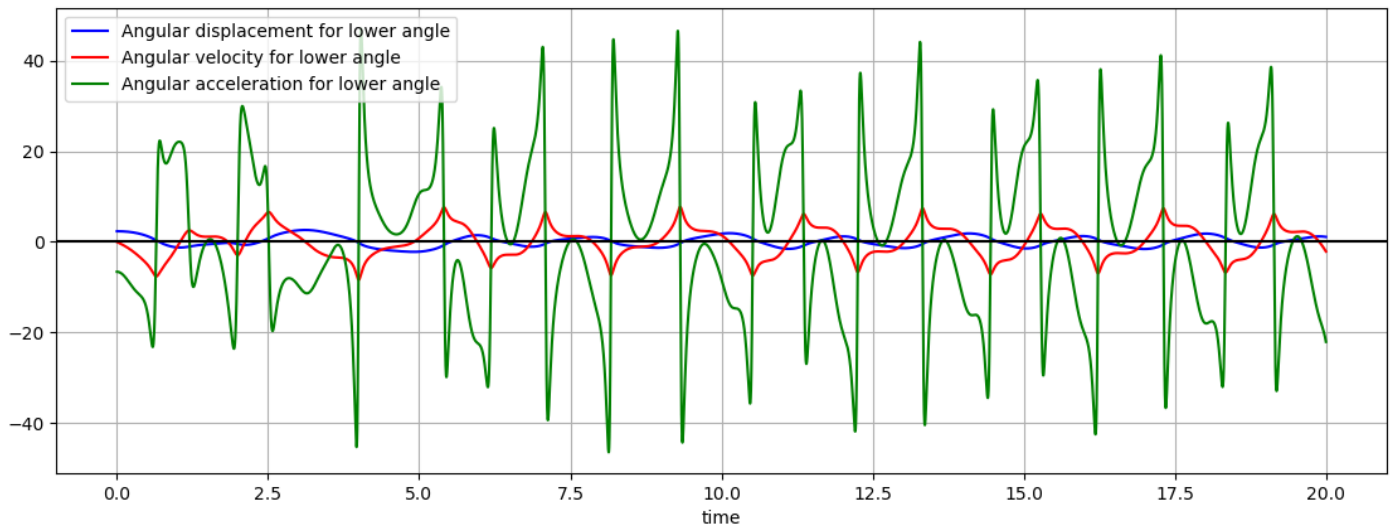


(b) Behaviour of  $\theta_2$

## A.7 Behaviour after 5% increase in $m_2$

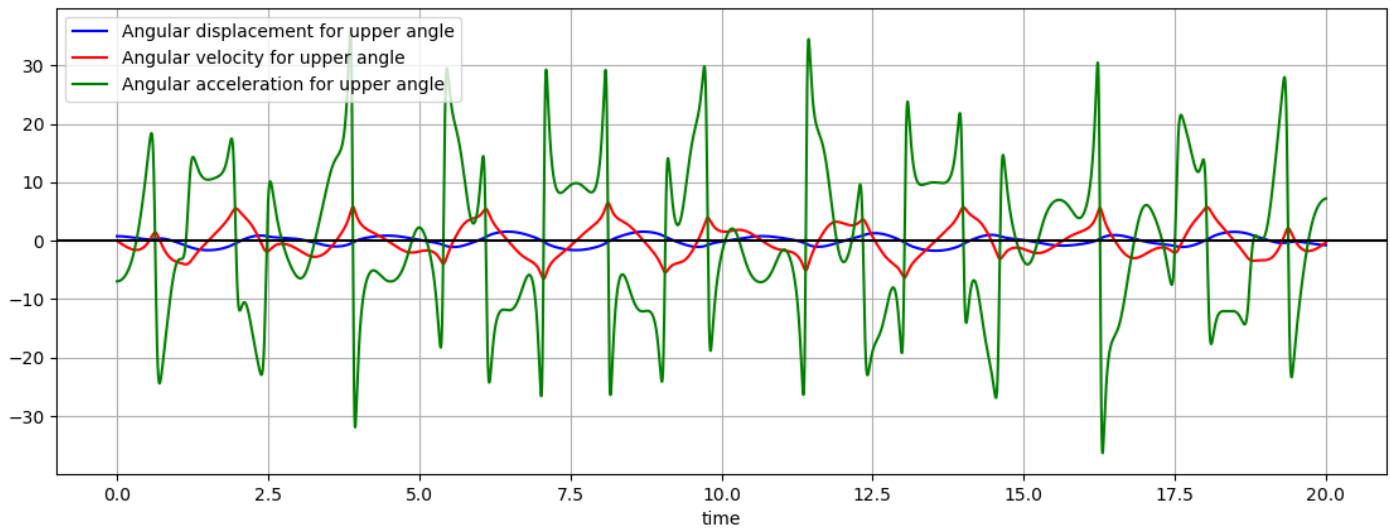


(a) Behaviour of  $\theta_1$

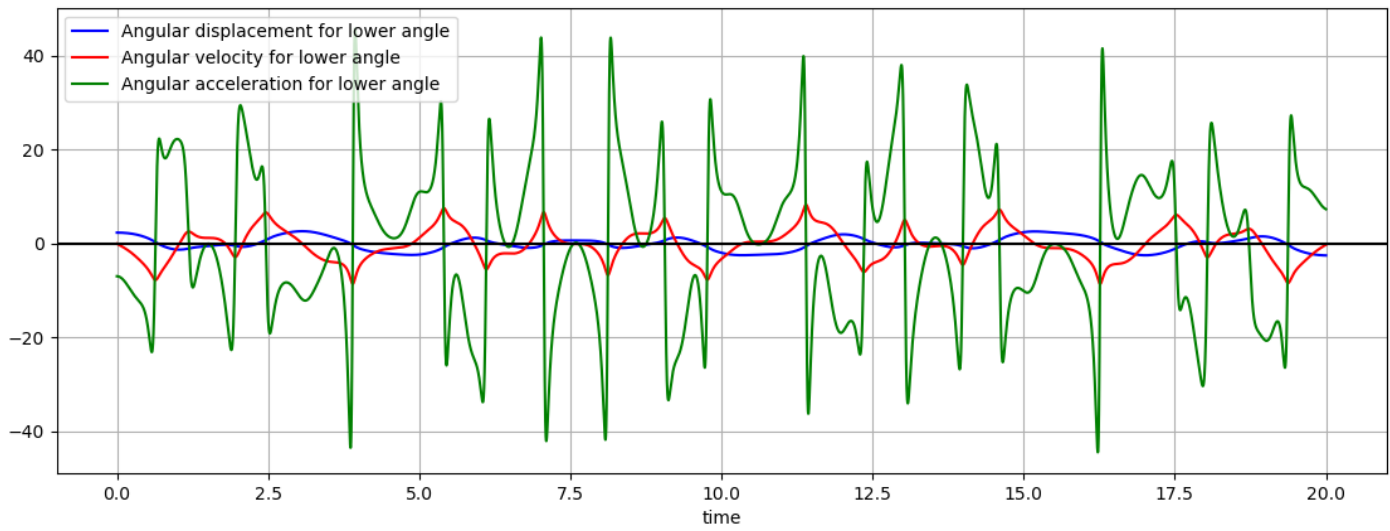


(b) Behaviour of  $\theta_2$

## A.8 Behaviour after 5% increase in $l_1$

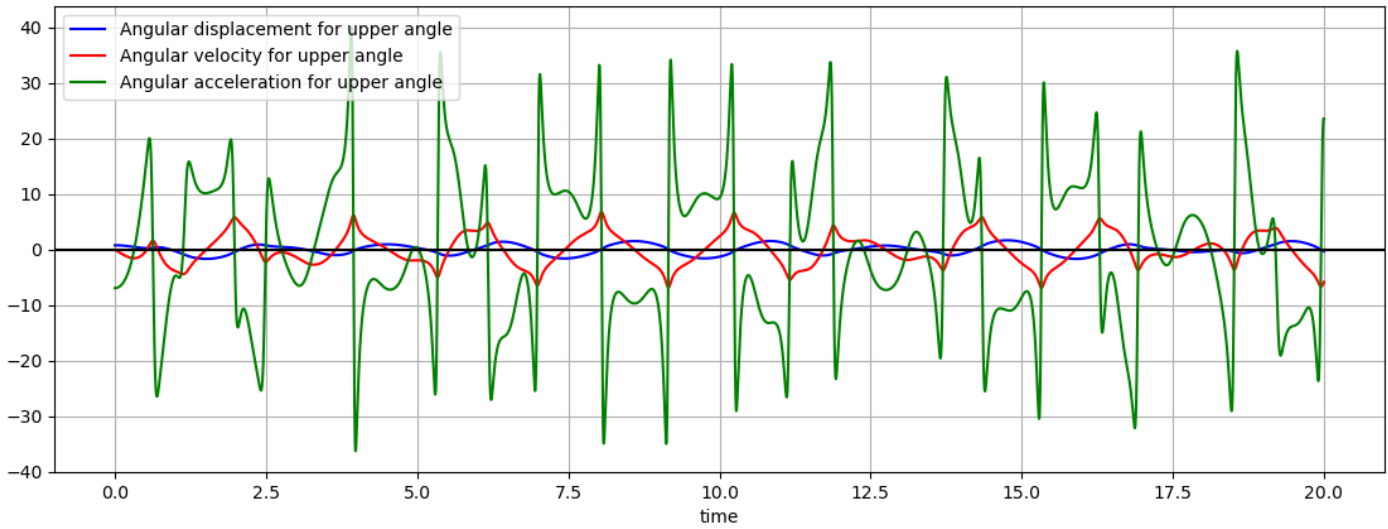


(a) Behaviour of  $\theta_1$

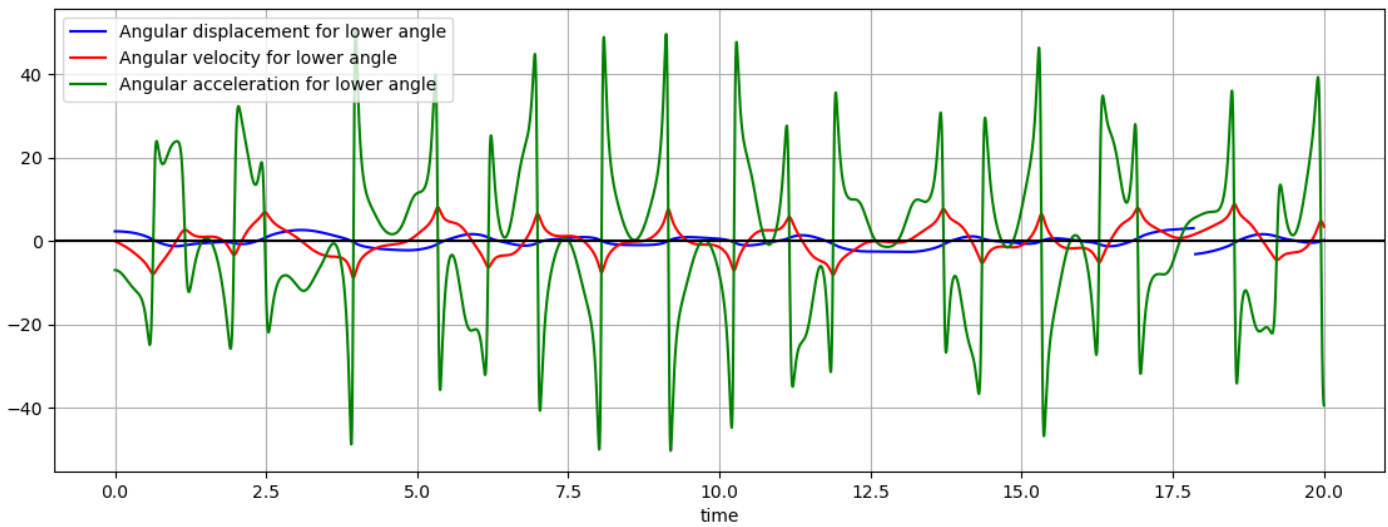


(b) Behaviour of  $\theta_2$

## A.9 Behaviour after 5% increase in $l_2$



(a) Behaviour of  $\theta_1$



(b) Behaviour of  $\theta_2$

## A.10 Approximation to single pendulum: small angle

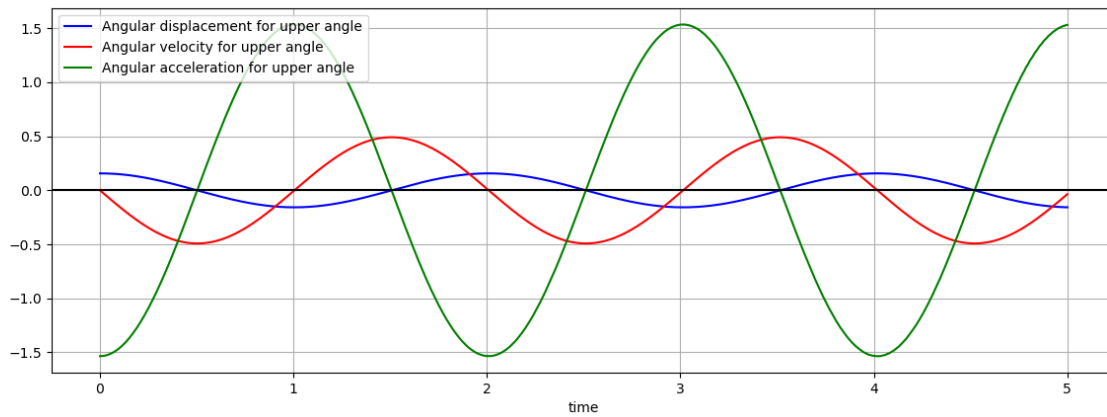


Figure 24: Behaviour of approximated single pendulum at low amplitude

## A.11 Approximation to double pendulum: large angle

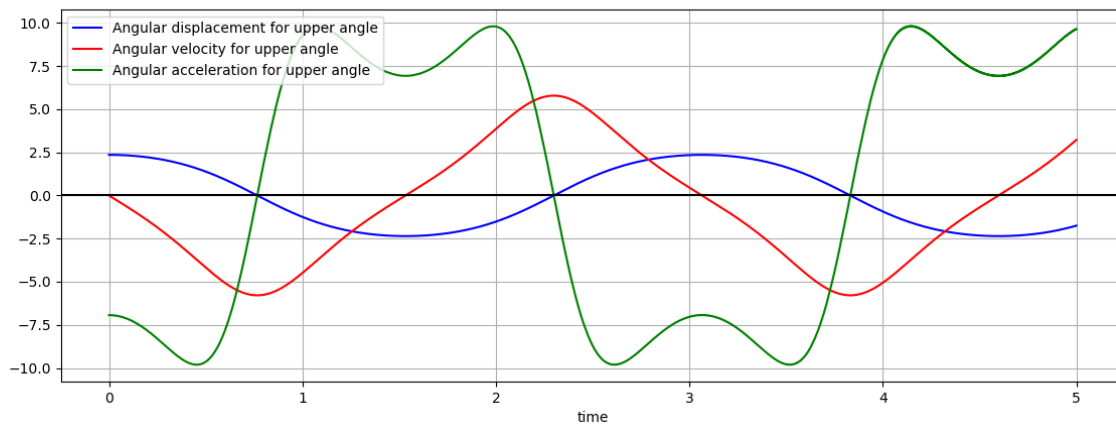


Figure 25: Behaviour of approximated single pendulum at high amplitude